.50 3.7 The Chain Rule

ctives

1) Use the chain rule to find the derivative of a composition $y = f(g(x))$
   a. Various notations all mean the same process
   b. Composition of three or more functions
   c. Combined with other derivative rules

Procedure for using the Chain Rule

Given a function $y = f(g(x))$ where *f* and *g* are differentiable.

Step 1: Identify the composition by finding an outer function *f* and an inner function *g*. Call $u = g(x)$.

Step 2: Use a *u*-substitution.

Replace $g(x)$ by u to express *y* in terms of *u*.

$y = f(g(x))$ becomes $y = f(u)$

Step 3: Find derivatives

$\frac{du}{dx}$ (by taking the derivative of the result from step 1, "the derivative of the inside") and

$\frac{dy}{du}$ (by taking the derivative of the result from step 2, "the derivative of the outside")

Step 4: Remove the *u*-substitution.

In $\frac{dy}{du}$ found in step 3 ("the derivative of the outside"), replace *u* by $g(x)$.

Step 5: Multiply $\frac{dy}{du} \cdot \frac{du}{dx}$ to get $\frac{dy}{dx}$

As you become proficient at the chain rule, you will be able to combine some of these steps and skip the *u*-sub.

Examples and Practice:

1) Given $f(x) = x^3$ and $g(x) = x^4 + 3x^2 - 5$,
   a. find $f(g(x))$.
   b. find $g(f(x))$
   c. Differentiate $f(g(x))$
   d. Differentiate $g(f(x))$

2) Given $f(x) = \tan x$ and $g(x) = x^2$, find
   a. $f(g(x))$.
   b. $g(f(x))$
   c. Differentiate $f(g(x))$
   d. Differentiate $g(f(x))$

3) Given $f(x)=x^3$, $g(x)=\sin x$ and $h(x)=4x$,

a. Find $f(g(h(x)))$

b. Differentiate $f(g(h(x)))$

4) Differentiate.

a. $f(x)=\sec(3x^2)$

b. $g(t)=\dfrac{-3}{(7t+1)^5}$

c. $f(x)=\sqrt[4]{(x^5+2)^3}$

d. $f(x)=x^4\sqrt{3-x^2}$

e. $f(t)=\dfrac{3t}{\sqrt[3]{5-t^2}}$

f. $g(x)=\dfrac{2\cot^2(7x)}{5}$

g. $y=\left(\dfrac{x^3-2}{4x+1}\right)^5$

h. $f(x)=\csc^3(5x^7+4x-1)$

i. $y=\cos(\sin(x^2))$

5) Find the equation of the tangent line to $y=\cos 3x$ at $\left(\dfrac{\pi}{4},-\dfrac{\sqrt{2}}{2}\right)$

6) Find the second derivative of $y=\sec^2 \pi x$

Basic Differentiation Formulas

In the table below, $u = f(x)$ and $v = g(x)$ represent differentiable functions of x

1. Derivative of a constant: $\frac{d}{dx}(c) = 0$
2. Derivative of a constant multiple:
$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$
3. Derivative of a sum or difference:
$\frac{d}{dx}[u \pm v] = \frac{du}{dx} \pm \frac{dv}{dx}$
4. Product Rule: $\frac{d}{dx}[u \cdot v] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$
5. Quotient Rule: $\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
6. Chain Rule: $\frac{d}{dx}[f(u)] = f'(u) \cdot \frac{du}{dx}$

General Formulas: With the chain rule . . .

$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}e^u = e^u\frac{du}{dx}$
$\frac{d}{dx}a^x = (\ln a)a^x$	$\frac{d}{dx}a^u = (\ln a)a^u\frac{du}{dx}$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}$
$\frac{d}{dx}(\cot x) = -\csc^2 x$	$\frac{d}{dx}(\cot u) = -\csc^2 u\frac{du}{dx}$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}(\sec u) = \sec u \tan u\frac{du}{dx}$
$\frac{d}{dx}(\csc x) = -\csc x \cot x$	$\frac{d}{dx}(\csc u) = -\csc u \cot u\frac{du}{dx}$

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<u>The Chain Rule</u> If f and g are differentiable and

If $y = f(g(x))$, call $u = g(x)$ so $y = f(u)$

then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

The Chain Rule

If $y = f(g(x))$

then $y'(x) = f'(\underbrace{g(x)}_{\text{Evaluate the outside function at the same inside function}}) \cdot \underbrace{g'(x)}_{\text{Multiply by derivative of the inside function.}}$

The Chain Rule

$$\frac{d}{dx}[f(g(x))] = (f \circ g)'(x)$$

$$= f'(g(x)) \cdot g'(x)$$

If several functions are composed, we must take derivatives of each function, working from outside in.

$$\frac{d}{dx}\left[f(g(h(k(x))))\right]$$

$$= f'(\underbrace{g(h(k(x)))}_{\text{Evaluate derivative at same composition used in original}}) \cdot g'(\underbrace{h(k(x))}_{\text{ditto}}) \cdot h'(\underbrace{k(x)}_{\text{ditto.}}) \cdot k'(x)$$

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① $f(x) = x^3$

$g(x) = x^4 + 3x^2 - 5$

a) $f(g(x)) = (g(x))^3$

$= \boxed{(x^4 + 3x^2 - 5)^3}$

this means

$(x^4+3x^2-5)(x^4+3x^2-5)(x^4+3x^2-5)$

ugh!

b) $g(f(x)) = (f(x))^4 + 3(f(x))^2 - 5$

$= (x^3)^4 + 3(x^3)^2 - 5$

$= \boxed{x^{12} + 3x^6 - 5}$

c) use chain rule to find $\frac{d}{dx}(x^4+3x^2-5)^3$

outer function $f(x) = x^3$

inner function $g(x) = x^4 + 3x^2 - 5$

$f(x) = x^3 \rightarrow f(u) = u^3$

$g(x) = u \rightarrow u = x^4 + 3x^2 - 5$

$\frac{du}{dx} = 4x^3 + 6x$

$\frac{df}{du} = 3u^2 \Longrightarrow 3(x^4+3x^2-5)^2$

$$\frac{d}{dx}(x^4+3x^2-5)^3 = \underbrace{3(x^4+3x^2-5)^2}_{\substack{\frac{df}{du} \\ \text{evaluated} \\ \text{at } u=g(x)}} \cdot \underbrace{(4x^3+6x)}_{\frac{du}{dx} = g'(x)}$$

Because we will want to solve the equation $f'(x) = 0$, the preferred final answer for derivatives is <u>fully</u> <u>factored</u> (not fully simplified.)

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$= 3(x^4+3x^2-5)^2 \cdot 2x(2x^2+3)$

b^2-4ac

$3^2-4(1)(-5)$

$9+20$

29 not a positive perfect square

not factorable

$= \boxed{6x(2x^2+3)(x^4+3x^2-5)^2}$

organize – – → →
with increasing complexity.

d) $\frac{d}{dx}(g(f(x))) = \frac{d}{dx}(x^{12}+3x^6-5)$

power rule + constant multiple rule + constant
no chain.

$= \boxed{12x^{11} + 18x^5}$

(2) $f(x) = \tan x$
$g(x) = x^2$

These are different!

a) $f(g(x)) = f(x^2) = \boxed{\tan(x^2)}$

b) $g(f(x)) = g(\tan x) = (\tan x)^2 = \boxed{\tan^2(x)}$

c) $\frac{d}{dx}(f(g(x))) = \frac{d}{dx}(\tan(x^2))$

outer function $\tan x \rightarrow \tan u$

inner function $u = x^2$

derivative of outer $\sec^2 u \rightarrow \sec^2(x^2)$

derivative of inner $2x$

multiply $\frac{d}{dx}(f(g(x))) = \boxed{2x \cdot \sec^2(x^2)}$

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d) $\frac{d}{dx}(\tan^2 x) = \frac{d}{dx}(\tan x)^2$

outer function $x^2 \to u^2$

inner function $u = \tan x$

derivative of outer $2u \to 2\tan x$

derivative of inner $\sec^2 x$

multiply $\frac{d}{dx}(g(f(x))) = \boxed{2\underbrace{\sec^2 x}_{\text{derivative inner}}\tan x}$

(3) $f(x) = x^3$

$g(x) = \sin x$

$h(x) = 4x$

a) find $f(g(h(x)))$

$= f(g(4x))$

$= f(\sin(4x))$

$= (\sin(4x))^3$

$= \boxed{\sin^3(4x)}$

b) $\frac{d}{dx} f(g(h(x))) = \frac{d}{dx}(\sin^3(4x)) = (\sin(4x))^3$

outer function $x^3 \to u^3$

middle function $u = \sin(4x) \to \sin w$

inner function $w = 4x$

derivative outer $= 3u^2 \to 3(\sin(4x))^2 \to 3\sin^2(4x)$

derivative middle $\cos w \to \cos(4x)$

derivative inner 4

multiply $4 \cdot \cos(4x) \cdot 3\sin^2(4x)$

$= \boxed{12\cos(4x)\sin^2(4x)}$

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(4) a) $f(x) = \sec(3x^2)$

outer $\sec(3x^2) \rightarrow \sec(u)$

inner $u = 3x^2$

diff outer $\sec(u)\tan(u) \rightarrow \sec(3x^2)\tan(3x^2)$

diff inner $6x$

mult $\boxed{f'(x) = 6x\sec(3x^2)\tan(3x^2)}$

b) $g(t) = \dfrac{-3}{(7t+1)^5}$

Rewrite! Now we can avoid the quotient rule. ☺

$g(t) = -3(7t+1)^{-5}$

outer $-3(7t+1)^{-5} \longrightarrow -3u^{-5}$

inner $u = 7t+1$

diff outer $-3(-5)u^{-5+1} = 15u^{-4} \longrightarrow 15(7t+1)^{-4}$

diff inner 7

multiply $7 \cdot 15(7t+1)^{-4}$

$$\boxed{g'(t) = \frac{105}{(7t+1)^4} = 105(7t+1)^{-4}}$$

c) $f(x) = \sqrt[4]{(x^5+2)^3}$

Rewrite! Use one chain rule instead of two.

$f(x) = (x^5+2)^{3/4}$

outer $f(x) = (x^5+2)^{3/4} \longrightarrow f(u) = u^{3/4}$

inner $u = x^5+2$

diff outer $\frac{3}{4}u^{3/4-1} = \frac{3}{4}u^{-1/4} \longrightarrow \frac{3}{4}(x^5+2)^{-1/4}$

diff inner $5x^4$

multiply $f'(x) = 5x^4 \cdot \dfrac{3}{4}(x^5+2)^{-1/4} = \boxed{\dfrac{15x^4}{4\sqrt[4]{x^5+2}}}$

CAUTION! order of op exp before mult means no dist

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d) $f(x) = x^4\sqrt{3-x^2}$

$\uparrow$

multiply $\Rightarrow$ product rule!

$$f'(x) = \left(\frac{d}{dx}x^4\right)\cdot\sqrt{3-x^2} + \left(\frac{d}{dx}\sqrt{3-x^2}\right)\cdot x^4$$

$$= 4x^3\sqrt{3-x^2} + \left(\text{chain rule!}\right)\cdot x^4$$

$$\frac{d}{dx}\sqrt{3-x^2} = \frac{d}{dx}(3-x^2)^{1/2}$$

outer $(3-x^2)^{1/2} \rightarrow u^{1/2}$

inner $3-x^2$

diff outer $\frac{1}{2}u^{1/2-1} = \frac{1}{2}u^{-1/2} \longrightarrow \frac{1}{2}(3-x^2)^{-1/2}$

diff inner $-2x$

mult $-2x\cdot\frac{1}{2}(3-x^2)^{-1/2}$

$$= \frac{-x}{\sqrt{3-x^2}}$$

$$f'(x) = 4x^3\sqrt{3-x^2} - \frac{x}{\sqrt{3-x^2}}\cdot x^4$$

$$f'(x) = 4x^3\sqrt{3-x^2} - \frac{x^5}{\sqrt{3-x^2}}$$

To factor fully, write with fractional exponents, factor least powers (GCF).

$$f'(x) = 4x^3(3-x^2)^{1/2} - x^5(3-x^2)^{-1/2}$$

x^3

$(3-x^2)^{-1/2}$

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$$f'(x) = x^3(3-x^2)^{-1/2}\left[\underbrace{\frac{4x^3(3-x^2)^{1/2}}{x^3(3-x^2)^{-1/2}}}_{\text{subtract exponents}} - \underbrace{\frac{x^5(3-x^2)^{-1/2}}{x^3(3-x^2)^{-1/2}}}_{\text{subtract exponents}}\right]$$

← This is the long view of what factoring does.

$$= x^3(3-x^2)^{-1/2}\left[4(3-x^2)^{1/2-(-1/2)} - x^{5-3}\right]$$

$$= x^3(3-x^2)^{-1/2}\left[4(3-x^2) - x^2\right]$$ $\quad \frac{1}{2}-(-\frac{1}{2}) = \frac{1}{2}+\frac{1}{2} = 1$

$$= x^3(3-x^2)^{-1/2}\left[12-4x^2-x^2\right]$$ distribute!

$$= x^3(3-x^2)^{-1/2}(12-5x^2)$$

$$\boxed{f'(x) = \frac{x^3(12-5x^2)}{\sqrt{3-x^2}}}$$

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e)

$$f(t) = \frac{3t}{\sqrt[3]{5-t^2}}$$

can be done by the Quotient Rule (see next page)

but better to rewrite with a negative exponent and use the product rule.

$$f(t) = 3t\,(5-t^2)^{-1/3}$$

product rule

$$f'(t) = 3t \cdot \frac{d}{dt}\left[(5-t^2)^{-1/3}\right] + \frac{d}{dt}[3t] \cdot (5-t^2)^{-1/3}$$

$$= 3t \cdot -\tfrac{1}{3}(5-t^2)^{-4/3} \cdot \underbrace{\frac{d}{dt}(5-t^2)}_{\text{chain rule}} + 3 \cdot (5-t^2)^{-1/3}$$

$$= 3t \cdot \frac{-1}{3}(5-t^2)^{-4/3} \cdot (-2t) + 3\,(5-t^2)^{-1/3}$$

$$= 2t^2(5-t^2)^{-4/3} + 3\,(5-t^2)^{-1/3}$$

OPTION 1: Write as denominators, find CD.

$$= \frac{2t^2}{(5-t^2)^{+4/3}} + \frac{3}{(5-t^2)^{+1/3}}$$

$$= \frac{2t^2}{(5-t^2)^{4/3}} + \frac{3(5-t^2)^{3/3}}{(5-t^2)^{1/3} \cdot (5-t^2)^{3/3}}$$

$$= \frac{2t^2 + 3(5-t^2)}{(5-t^2)^{4/3}}$$

$$= \frac{2t^2 + 15 - 3t^2}{(5-t^2)^{4/3}}$$

$$= \boxed{\frac{15-t^2}{(5-t^2)^{4/3}}}$$

OPTION 2: Factor out least powers $(-4/3$ less than $-1/3)$

$$= (5-t^2)^{-4/3} \cdot \left[2t^2 + 3(5-t^2)^{-\frac{1}{3}-\left(-\frac{4}{3}\right)}\right]$$

Subtract exp being factored out

$$= (5-t^2)^{-4/3}\left[2t^2 + 3(5-t^2)\right]$$

$$= \frac{2t^2 + 15 - 3t^2}{(5-t^2)^{4/3}}$$

$$= \boxed{\frac{15-t^2}{(5-t^2)^{4/3}}}$$

Method 2:

$$f(t) = \frac{3t}{\sqrt[3]{5-t^2}}$$

OR better! re-write

$f(t) = 3t(5-t^2)^{-1/3}$ - see previous page

$$f'(t) = \frac{\sqrt[3]{5-t^2} \cdot \frac{d}{dt}(3t) - 3t \cdot \frac{d}{dt}\left(\sqrt[3]{5-t^2}\right)}{\left(\sqrt[3]{5-t^2}\right)^2}$$ Quotient Rule

$$= \frac{(5-t^2)^{1/3} \cdot 3 - 3t \cdot \frac{1}{3}(5-t^2)^{-2/3} \cdot (-2t)}{(5-t^2)^{2/3}}$$ Chain Rule

$$= \frac{3(5-t^2)^{1/3} + 2t^2(5-t^2)^{-2/3}}{(5-t^2)^{2/3}}$$ Factor out least power

$$= (5-t^2)^{-2/3}\left\{\frac{3(5-t^2)^{1/3-(-2/3)} + 2t^2}{(5-t^2)^{2/3}}\right\}$$ Move neg exp to denom

$$= \frac{3(5-t^2) + 2t^2}{(5-t^2)^{4/3}}$$ Dist

$$= \frac{15 - 3t^2 + 2t^2}{(5-t^2)^{4/3}}$$ Combine

$$= \boxed{\frac{15-t^2}{\sqrt[3]{(5-t^2)^4}}} \text{ or } \boxed{\frac{-(t^2-15)}{\sqrt[3]{(5-t^2)^4}}}$$ Write as radical

f) $g(x) = \frac{2\cot^2(7x)}{5}$

chain rule 3-way

Rewrite! $g(x) = \frac{2}{5}\cot^2(7x)$

constant multiple

$g(x) = \frac{2}{5}\left[\cot(7x)\right]^2$

outer $u^2 \xrightarrow{\text{diff}} 2u$

middle $\cot w \xrightarrow{\text{diff}} -\csc^2 w$

inner $7x \xrightarrow{\text{diff}} 7$

$$g'(x) = \underbrace{\frac{2}{5}}_{\text{constant mult}} \cdot \underbrace{2\left(\cot(7x)\right)^1}_{\text{diff outer}} \cdot \underbrace{\left(-\csc^2(7x)\right)}_{\text{diff middle}} \cdot \underbrace{7}_{\text{diff inner}}$$

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$$g'(x) = \frac{2}{5} \cdot 2 \cdot (-1) \cdot 7 \cdot \cot(7x) \cdot \csc^2(7x)$$

$$= \boxed{\frac{-28}{5} \csc^2(7x) \cot(7x)}$$

g) $y = \left(\frac{x^3-2}{4x+1}\right)^5$ chain rule $(\quad)^5$ u^5 outer $\xrightarrow{\text{diff}}$ $5u^4$

inner: $\left(\frac{x^3-2}{4x+1}\right)$ quotient rule! diff inner

$$y' = \underbrace{5\left(\frac{x^3-2}{4x+1}\right)^4}_{\text{diff outer}} \cdot \underbrace{\left[\frac{(4x+1)\cdot\frac{d}{dx}(x^3-2) - (x^3-2)\cdot\frac{d}{dx}(4x+1)}{(4x+1)^2}\right]}_{\text{diff inner}}$$

$$y' = \underbrace{5\,\frac{(x^3-2)^4}{(4x+1)^4}}_{\text{separate}} \cdot \left[\frac{(4x+1)\cdot(3x^2) - (x^3-2)\cdot 4}{(4x+1)^2}\right]$$

$\left(\frac{a}{b}\right)^4 = \frac{a^4}{b^4}$

$$y' = \frac{5(x^3-2)^4\left[12x^3 + 3x^2 - 4x^3 + 8\right]}{(4x+1)^6}$$

dist $3x^2$ 1st term
dist (-4) 2nd term

add exp

$$\boxed{y' = \frac{5(x^3-2)^4(8x^3+3x^2+8)}{(4x+1)^6}}$$

remember this exponent is outermost

h) $f(x) = \csc^3(5x^7+4x-1)$

$f(x) = \underbrace{\left[\csc(5x^7+4x-1)\right]^3}_{\text{chain rule twice}}$

outer $u^3 \xrightarrow{\text{diff}} 3u^2$

middle $\csc w \xrightarrow{\text{diff}} -\csc w \cot w$

inner $5x^7+4x-1 \xrightarrow{\text{diff}} 35x^6+4$

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$$f'(x) = \underbrace{3\left(\csc\left(5x^7+4x-1\right)\right)^2}_{\text{diff outer}} \cdot \underbrace{\left(-\csc\left(5x^7+4x-1\right)\cdot\cot\left(5x^7+4x-1\right)\right)}_{\text{diff middle}}$$

$$\cdot \underbrace{\left(35x^6+4\right)}_{\text{diff inner}}$$

$$\boxed{f'(x) = -3\left(35x^6+4\right)\csc^3\left(5x^7+4x-1\right)\cot\left(5x^7+4x-1\right)}$$

add exp

$\csc^2(\text{same})\cdot\csc(\text{same})$

i) $y = \cos(\sin(x^2))$ chain rule twice

outer $\cos u \xrightarrow{\text{diff}} -\sin u$

middle $\sin w \xrightarrow{\text{diff}} \cos w$

inner $x^2 \xrightarrow{\text{diff}} 2x$

$$y' = \underbrace{-\sin(\sin(x^2))}_{\text{outer}} \cdot \underbrace{\cos(x^2)}_{\text{middle}} \cdot \underbrace{2x}_{\text{inner}}$$

$$\boxed{y' = -2x\cdot\cos(x^2)\cdot\sin(\sin(x^2))}$$

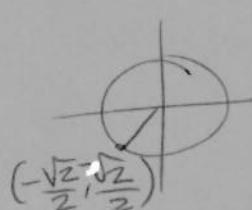
(5) eqn of tangent to $y=\cos(3x)$ at $\left(\frac{\pi}{4}, -\frac{\sqrt{2}}{2}\right)$

slope of tangent $y'\left(\frac{\pi}{4}\right)$

$$y'(x) = \underbrace{-\sin(3x)}_{\text{outer}} \cdot \underbrace{3}_{\text{inner}} = -3\sin(3x)$$

$$y'\left(\frac{\pi}{4}\right) = -3\sin\left(3\cdot\frac{\pi}{4}\right)$$

$$= -3\sin\left(\frac{3\pi}{4}\right)$$



$$= -3\cdot\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2} = m_{\tan}$$

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equation of tangent

$$y - \left(-\frac{\sqrt{2}}{2}\right) = \frac{3\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right)$$

$$y + \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}x - \frac{3\sqrt{2}\pi}{2\cdot 4}$$

$$y = \frac{3\sqrt{2}}{2}x - \frac{3\sqrt{2}\pi}{8} - \frac{\sqrt{2}\cdot 4}{2\cdot 4}$$

$$\boxed{y = \frac{3\sqrt{2}x}{2} - \left(\frac{3\sqrt{2}\pi + 4\sqrt{2}}{8}\right)}$$

or even

$$\boxed{y = \frac{3\sqrt{2}}{2}x - \frac{\sqrt{2}(3\pi + 4)}{8}}$$

6 y'' if $y = \sec^2(\pi x) = [\sec(\pi x)]^2$ chain rule twice

$$y' = \underbrace{2(\sec \pi x)}_{\substack{\text{outer} \\ u^2}} \cdot \underbrace{(\sec \pi x \tan \pi x)}_{\substack{\text{middle} \\ \sec w}} \cdot \underbrace{\pi}_{\substack{\text{inner} \\ \pi x}}$$

$$y' = \underbrace{2\pi}_{\substack{\uparrow \\ \text{constant} \\ \text{multiple}}} \cdot \underbrace{\underbrace{\sec^2 \pi x}_{\text{chain rule 2x}} \cdot \underbrace{\tan \pi x}_{\text{chain 1x}}}_{\text{product rule}}$$

$$y'' = 2\pi\left[\underbrace{\frac{d}{dx}(\sec^2 \pi x)}_{\text{chain 2x}} \cdot \tan \pi x + (\sec^2 \pi x) \cdot \underbrace{\frac{d}{dx}(\tan \pi x)}_{\text{chain 1x}}\right]$$

outer $u^2 \xrightarrow{\text{diff}} 2u$
middle $\sec w \xrightarrow{\text{diff}} \sec w \tan w$
inner $\pi x \xrightarrow{\text{diff}} \pi$

outer $\tan u \xrightarrow{\text{diff}} \sec^2 u$
inner $\pi x \xrightarrow{\text{diff}} \pi$

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$$y'' = 2\pi\left[\left(2\sec\pi x \cdot \sec\pi x\tan\pi x \cdot \pi\right)\cdot \tan\pi x + \sec^2\pi x \cdot \left(\sec^2\pi x\right)\cdot\pi\right]$$

$$y'' = 2\pi\left[2\pi\sec^2\pi x\tan^2\pi x + \pi\sec^4\pi x\right]$$ GCF $\pi\sec^2\pi x$

$$\boxed{y'' = 2\pi^2\sec^2\pi x\left(2\tan^2\pi x + \sec^2\pi x\right)}$$

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Extras

Differentiate

(A) $g(t) = \sqrt{\sqrt{t+1}+1}$

step 1: Rewrite w/ exponents, identify inside & outside

$$g(t) = \left((t+1)^{1/2} + 1 \right)^{1/2}$$

Step 2: Differentiate using chain rule.

$$g'(t) = \tfrac{1}{2}\left[(t+1)^{1/2} + 1 \right]^{-1/2} \cdot \frac{d}{dx}\left[(t+1)^{1/2} + 1 \right]$$

$$= \tfrac{1}{2}\left[(t+1)^{1/2} + 1 \right]^{-1/2} \cdot \tfrac{1}{2}(t+1)^{-1/2} \cdot \frac{d}{dt}(t+1) + 0$$

$$= \frac{1}{2}\left[(t+1)^{1/2} + 1 \right]^{-1/2} \cdot \frac{1}{2}(t+1)^{-1/2} \cdot 1$$

$$\boxed{g'(t) = \frac{1}{4\sqrt{\sqrt{t+1}+1} \cdot \sqrt{t+1}}}$$

(B) $g(x) = \left(2 + (x^2+1)^4\right)^3$

$$g'(x) = 3\left(2 + (x^2+1)^4\right)^2 \cdot \frac{d}{dx}\left[2 + (x^2+1)^4\right]$$ chain rule twice

$$= 3\left(2 + (x^2+1)^4\right)^2 \cdot 4(x^2+1)^3 \cdot \frac{d}{dx}(x^2+1)$$

$$= 3\left(2 + (x^2+1)^4\right)^2 \cdot 4(x^2+1)^3 \cdot 2x$$

$$\boxed{g'(x) = 24x(x^2+1)^3\left[2 + (x^2+1)^4\right]^2}$$

(C) $g(x) = \left(\dfrac{3x^2-2}{2x+3}\right)^3$

$$g'(x) = 3\left(\frac{3x^2-2}{2x+3}\right)^2 \cdot \frac{d}{dx}\left[\frac{3x^2-2}{2x+3}\right]$$ chain rule

$$= 3\left(\frac{3x^2-2}{2x+3}\right)^2 \cdot \frac{(2x+3)\cdot 6x - (3x^2-2)\cdot 2}{(2x+3)^2}$$ quotient rule

$$= \frac{3(3x^2-2)^2(12x^2+18x-6x^2+4)}{(2x+3)^4}$$ distribute

$$= \frac{3(3x^2-2)^2(6x^2+18x+4)}{(2x+3)^4}$$

$$\boxed{g'(x) = \frac{6(3x^2-2)^2(3x^2+9x+2)}{(2x+3)^4}}$$

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(D) $y = \dfrac{x}{\sqrt{x^4+4}}$

rewrite

$y = x(x^4+4)^{-1/2}$

product

$y' = x \cdot -\frac{1}{2}(x^4+4)^{-3/2}\underbrace{(4x^3)}_{\text{chain}} + 1\cdot(x^4+4)^{-1/2}$

$y' = -2x^4(x^4+4)^{-3/2} + (x^4+4)^{-1/2}$

factor least powers (see p7)

$y' = (x^4+4)^{-3/2}\left[-2x^4 + (x^4+4)^{-1/2-(-3/2)}\right]$

$= \dfrac{-2x^4+x^4+4}{(x^4+4)^{3/2}}$

$= \boxed{\dfrac{-x^4+4}{(x^4+4)^{3/2}}}$

(E) $y = \frac{1}{2}x^2\sqrt{16-x^2}$

product rule

$y' = \frac{1}{2}x^2 \cdot \frac{1}{2}(16-x^2)^{-1/2}\underbrace{(-2x)}_{\text{chain}} + x(16-x^2)^{1/2}$

$y' = -\frac{1}{2}x^3(16-x^2)^{-1/2} + x(16-x^2)^{1/2}$

factor least powers + LCD

$y' = -\frac{x}{2}(16-x^2)^{-1/2}\{+x^2 - 2(6-x^2)\}$

$= \dfrac{-x(x^2-12+2x^2)}{2(16-x^2)^{1/2}}$

$= \dfrac{-x(3x^2-12)}{2(16-x^2)^{1/2}}$

$= \dfrac{-3x(x^2-4)}{2(16-x^2)^{1/2}}$

$= \boxed{\dfrac{-3x(x-2)(x+2)}{2(4-x)^{1/2}(4+x)^{1/2}}}$

(24) $y = \dfrac{-5}{(t+3)^3}$

rewrite

$y = -5(t+3)^{-3}$

chain rule not necessary

$y' = -5\cdot 3(t+3)^{-4}\cdot 1$

$\boxed{y' = \dfrac{-15}{(t+3)^4}}$

(25) $s(t) = \dfrac{1}{t^2+3t-1}$

rewrite

$s(t) = (t^2+3t-1)^{-1}$

chain rule

$s'(t) = -(t^2+3t-1)^{-2}\cdot(2t+3)$

$\boxed{s'(t) = \dfrac{-(2t+3)}{(t^2+3t-1)^2}}$

(26) $g(x) = \sqrt{x^2-2x+1}$

rewrite $g(x) = (x^2-2x+1)^{1/2}$

chain rule

$g'(x) = \frac{1}{2}(x^2-2x+1)^{-1/2}\cdot(2x-2)$

$= \dfrac{\frac{1}{2}\cdot 2(x-1)}{(x^2-2x+1)^{1/2}}$

$\boxed{g'(x) = \dfrac{x-1}{\sqrt{x^2-2x+1}}}$

(27) $f(t) = (9t+2)^{2/3}$

chain rule

$f'(t) = \frac{2}{3}(9t+2)^{-1/3}\cdot 9$

$= \dfrac{2\cdot 3}{(9t+2)^{1/3}}$

$\boxed{f'(t) = \dfrac{6}{(9t+2)^{1/3}}}$

(28) $y = 2(6-x^2)^5$

$y' = 2\cdot 5(6-x^2)^4(-2x)$

$\boxed{y' = -20x(6-x^2)^4}$

Alternate approach to chain rule using u-substitution:

(F) $f(x) = \sec(5x^2 - \sqrt[3]{x}) = g(u(x))$

inside function $u(x) = 5x^2 - \sqrt[3]{x}$ as a function of x

outside function $\sec(u)$ as a function of u

$$f'(x) = \frac{d}{du}[\sec u] \cdot \frac{d}{dx}[u(x)]$$
$$= \sec u \cdot \tan u \cdot u'(x)$$

$$\begin{cases} u = 5x^2 - x^{1/3} \\ u'(x) = 10x - \frac{1}{3}x^{-2/3} \end{cases}$$

substitute for u and u':

$$= \boxed{\sec(5x^2 - \sqrt[3]{x})\tan(5x^2 - \sqrt[3]{x}) \cdot (10x - \tfrac{1}{3}x^{-2/3})}$$

An alternate approach:

(G) $f(x) = \left(\frac{x-5}{2x+1}\right)^3 = \frac{(x-5)^3}{(2x+1)^3}$

quotient rule first

$$f'(x) = \frac{(2x+1)^3 \cdot \frac{d}{dx}[(x-5)^3] - (x-5)^3 \cdot \frac{d}{dx}[(2x+1)]^3}{[(2x+1)^2]^3}$$

$$= \frac{(2x+1)^3 \cdot 3(x-5)^2 \cdot 1 - (x-5)^3 \cdot 3(2x+1)^2 \cdot 2}{(2x+1)^6}$$

factor least powers $(2x+1)^2$ and $(x-5)^2$:

$$= \frac{(2x+1)^2(x-5)^2[(2x+1) \cdot 3 - (x-5) \cdot 6]}{(2x+1)^6}$$

reduce exp on $(2x+1)$; dist and combine

$$= \frac{(x-5)^2[6x+3-6x+30]}{(2x+1)^4}$$

$$= \boxed{\frac{33(x-5)^2}{(2x+1)^4}}$$